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Analysis of Dirac Operators with interactions Supported on the boundaries of rough domains

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Abstract

We consider the Dirac operator $H = \alpha D + F(x)$ on the Hilbert space $L^2(\mathbb{R}^2, \mathbb{C}^2)$ Where it turns into the phenomenon of contraction, where $F(x)$ is a 2×2 Hermitian matrix valued function which decays suitable for infinity. We studied Dirac operators in terms of semi-classical differential operators, where the semi-classical phenomenon is given by the mass inverse. We show that the zero resonance is absent for H, extending recent results of Dirac operators are semi-classical sparse differential operators with an order that leads to zero, and we show their main semi-classical forms. Then, using some regularity properties of the operator, we show that the. We will study the Dirac factor in relation to aspects that depend on additional mass outside the normal field. Then the additional mass is very large. Using calculus and the basic properties of Dirac operators, we construct a property that the oscillating working solvent shares with the oscillating working solvent. We will show that the oscillatory operator converges in a standard solvent manner toward the operator and gives convergence and a sharp estimate of the rate of convergence.

Keywords: Dirac operators, zero resonances, Hilbert space, Riemannian manifold, spinors, Newman

*** Introduction**

This study is devoted to the spectral of two types of perturbation of the Dirac operator, which are singular from the point of view of scaling. The coupling of the Dirac operator with a set of delta frame interactions of electrostatic, numerical Lorentz and magnetic type is studied. either on regular compact surfaces or locally deformed hyperplanes. [1,2] We develop an approach based on regularization techniques that will allow us to describe the self-adjoint realization of the perturbed Dirac operator for any combination of the coupling constants. We will confirm by verifying the basic spectroscopic properties of the heterogeneous models using Berman-Schwinger's principle and the formula in which the solvent of the oscillating agent is coupled to the solvent of the oscillating agent. Free Dirac operator: We are particularly interested in the case of critical models of function constants that lead to the phenomenon of contraction, functions of the Dirac operator with non-critical models of delta interactions supported on irregular surfaces.[3] We will generalize the results first obtained in the context of regular surfaces to the case of surfaces locally coincident with the function whose gradient is bounded and has vanishing mean oscillations. We use some techniques from harmonic and Numerical analysis, potential theory and Fredholm's theory. Moreover, we show how the smoothness of the surface supporting the delta interactions affects the domain of the operator under consideration. We investigate delta-interactions supported on surfaces satisfying certain weak topological conditions. [4,5] We study the Dirac operator coupled on uniformly rectifiable surfaces. Under certain conditions on the coupling constants, we prove the self-jointness of the perturbed operator and we establish several spectral properties in the Lipschitz case. We will define the fundamental spectrum of the oscillatory operator and show that it can appear at most in a limited number of eigenvalues at the fundamental depth of the operator. Moreover, we result to other delta-shell interactions and derive several models of Dirac operators that give rise to the confinement phenomenon. We are concerned with the study of the pseudodifferential properties of operators associated with the Dirac operator with the boundary condition.[6]

*** Definitions**

Suppose that we are given some definitions of a compact oriented Riemannian manifold (N, δ) a spinor module with conjugation (K, C), together with a Riemannian metric μ, so that the Clifford action $c: B \rightarrow End_A(K)$ has been specified.[7] We can be also writing it as $c \in Hom_A(B \otimes K, K)$ by setting $c(k \otimes \varphi) = c(k)\varphi$.

*** Definitions**

Using the inclusion of $A^1(N) \rightarrow B$ where in the odd dimensional case this is given by $c(\beta) = c(\beta\omega)$, as before we can be forming the composition as:

$$D = ic \circ \nabla^K \quad (1)$$

Where:

$$K \xrightarrow{\nabla^K} A^1(N) \bigotimes_A K \xrightarrow{c} K$$

So that $D: K \longrightarrow K$ is $\mathbb{C}$ linear. This is the Dirac operator Association to (K, C) *and* μ.

The $(-i)$ is included in the definition to make D symmetric as an operator on Hilbert space, because we have chosen μ to be positive definite, that is, $\omega^\alpha\omega^\beta + \omega^\beta\omega^\alpha = +2\delta^{\alpha\beta}$.

Historically, D was introduced as $-i\omega^\kappa\delta_\kappa = \omega^\kappa p_\kappa$ where the p_κ are components of a $4\, momentum$. [7,8] Using local bases for $\mathfrak{B}(N)$ and $A^1(N)$, we get nicer formulas

$$D\varphi = -ic\nabla^K_{\partial_j}\varphi = -i\omega^\alpha\nabla^K_{E_\alpha}\omega. \quad (2)$$

The essential algebraic properties of D is the commutation relation:

$$[D, \alpha] = -ic(da), \;\; \forall a \in A = C^{+\infty}(N) \quad (3)$$

And indeed

$$[D, \alpha]\varphi = -i\hat{c}\big(\nabla^K(a\varphi)\big) + ia\hat{c}\,(\nabla^K\varphi) = -i\hat{c}(\nabla^K(a\varphi) - a\nabla^K\varphi) = -i\hat{c}(da \otimes \varphi) = -ic(da)\varphi, \;\; \forall\, \varphi \in K \quad (4)$$

3. the metric distance property and operator

As an operator, we can realize of $[D, \alpha]$ by giving K the structure of a Hilbert space, If we can write the determinants $\mu = det(\mu_{ij})$ concisely, we have:

$$u_\mu = \sqrt{det\mu}\, dx^1 \wedge dx^2 \wedge \ldots \wedge dx^n \in A^n(N) \quad (5)$$

And the inner product is:

$$\langle \emptyset | \varphi \rangle = \int_N (\emptyset|\varphi)\, u_\mu, \quad \forall \emptyset, \varphi \in K. \tag{6}$$

On completion of the norm $\|\varphi\| = \sqrt{\langle \varphi, \varphi \rangle}$, we get the Hilbert space $\mathcal{H} = L^2(N, K)$ of L^2- spinors on N.

Using the gradient in shot $grada = (da) \in \mathcal{X}$, we can compute.

$$\|D, a\|^2 = \|c(aa)\|^2$$
$$= \sup_{x \in N} \left\| c_x\big(da(x)\big) \right\|^2$$
$$= \sup_{x \in N} \mu_x\big(d\bar{a}(x), da(x)\big) \text{ and } \mu_x \text{ on } (T_x^* N)^{\mathbb{C}}$$
$$= \sup_{x \in N} \mu_x(grad\bar{a}|_x, grada|_x), \text{ using the dual } \mu_x \text{ on } (T_x^* N)^{\mathbb{C}}$$
$$= \sup_{x \in N} \| grada|_x \|^2 = \| grada|_x \|_\infty^2.$$

We can calculate the distances on a Riemannian manifold with the following formula:

$$d(x, y) = inf\{length(\eta) : \eta : [0,1] \to N,\ \eta(0) = x,\ \eta(1) = y\}, \tag{7}$$

With use on all smooth multipart paths η in the form ω in N For a x to y. [9]

For $a \in C^\infty(N)$ we obtain:

$$a(y) - a(x) = a\big(\eta(1)\big) - a\big(\eta(0)\big)$$
$$= \int_0^1 \frac{d}{dt}\Big(a\big(\eta(t)\big)\Big)\, dt$$
$$= \int_0^1 \eta(0)|_{\eta(t)} dt$$
$$= \int_0^1 da(\eta)|_{\eta(t)} dt$$
$$= \int_0^1 da_{\eta(t)}\big(\eta(t)\big) dt$$
$$= \int_0^1 \mu_{\eta(t)}\Big(grad\ a|_{\eta(t)}, \eta(t)\Big)\, dt, \tag{8}$$

and we can estimate this difference by:

$$|a(y) - a(x)| \le \int_0^1 |grada|_{\varphi(t)} |\eta(t)|\, dt \le \|grad\ a\|_\infty \int_0^1 |\varphi(t)|\, dt = \|grad\ a\|_\infty\, length(\eta) = \|[D, a]\|\, length(\eta)$$

Thus

$$sup\{|a(y) - a(x)| : a \in C(N), \|[D, a]\| \le 1\} \le \inf length(\varphi) = d(x, y) \tag{10}$$

In this supremum we can use $a \in C(N)$, a need only be continuous with grad a bounded. Since we have obtained $|a(y) - a(x)| \le \|grad\ a\|_\infty d(x, y)$, we see that a need only be Lipschitz on $N -$ with respect to the distance $d -$ with Lipschitz $constant \le \|grad\ a\|_\infty$.[10,11,12]

In fact, this is the best general Lipschitz constant: $fix\ x \in N$, and set $a_x(y) = d(x,y)$. This function lies in $C(N)$, and $|a_x(y) - a_x(z)| \leq d(y,z)$, by the triangle.

Inequality for d. Since $\|grad\ a_x\|_\infty = 1$ by a local geodesic calculation, we see that $a = a_x$ makes the inequality in (10) sharp:

$$d(x,y) = sup\{|a(y) - a(x)|: \|grad\ a_x\|_\infty = 1\}$$
$$= sup\{|a(y) - a(x)|: a \in C(N), \|[D,a]\| \leq 1\} \qquad (11)$$

So that D determines the Riemannian distance d, which in turn determines the metric μ.

Example 3.1. Take $N = \mathbb{P}^1$, $(n = 1, m = 0,\ 2^m = 1)$. The trivial line bundle is a spinor bundle, with $K = C^\infty(\mathbb{K}^1) = A$, and C is just the complex conjugation K of functions. [13] With the flat metric on $\mathbb{P}^1 \cong \frac{\mathbb{R}}{\mathbb{Z}}$, we can identify P with the set of smooth $1-$ periodic functions on $\mathbb{R}$, so that both ∇ and ∇^P are trivial since $\Gamma_{11}^1 = 0$. [13] Therefore, we have:

$$D = -\frac{d\beta}{d\theta}$$

is the Dirac operator in this case. Thus $[D,f] = -if'$ for $f \in \mathcal{A}$, and for $\alpha, \beta \in [0,1]$, we have:

$$|f(\beta) - f(\alpha)| = \left|\int_\alpha^\beta f'(\theta)d\theta\right|$$
$$\leq \int_\alpha^\beta |f'(\theta)|\, d\theta$$
$$\leq |\beta - \alpha| \qquad (12)$$

Whenever $\|f'\|_\infty \leq 1$

Using $f_\alpha(\beta) = |\beta - \alpha|$ for $\alpha - \frac{1}{2} \leq \beta \leq \alpha + \frac{1}{2}$ wrapped around $\frac{\mathbb{R}}{\mathbb{Z}}$, we get a Lipschitz function making the inequality shar. So $d(\alpha,\beta) = |\beta - \alpha|$ Given that $|\beta - \alpha| \leq \frac{1}{2}$, this is the length of the arc on circumference 1. [14] In general, the formula $d(x,y)$ gives the length of the geodesic minimum from x to y, provided y converges to x, then the of x.

*** Symmetry of the Dirac operator**

We regard D as an operator on $L^2(N,S)$, defined initially on domain $S = \Gamma(N,S)$. And D is symmetric and synonymous with that is, whenever $\alpha, \beta \in S$, the following equality holds:

$$\langle D\alpha|\beta\rangle = \langle \alpha|D\beta\rangle \qquad (13)$$

Proof: we compute the pairings $(D\alpha|\beta)$ and $(\alpha|D\beta)$, which take value in $B = \mathbb{C}^\infty(N)$. We need a formula the divergence of a vector field $L_X u_g = (div\ X)u_g$, so that:

$$\int_N (div\,X)u_g$$

$$= \int_N L_X(u_g)$$

$$= \int_N l_X(du_g) + d(l_X u_g)$$

$$= \int_N d(l_X u)$$

$$= 0 \qquad (14)$$

By Stokes theorem (remember that N has no boundary). [14] This formula is:

$$divX = \delta_j X^j + \Gamma_{jk}^j x^k$$
$$= dx^j \left(gred_{\delta_j} X\right) \qquad (15)$$

As can be seen on the right hand side we use the Levi-Civita connection on $X(N)$. Now we abbreviated $c^j = c(dx^j) \in \Gamma(U, endS), \quad for\ j = 1, \dots, n$. [8] Then we compute the difference of $B - valued$ pairings:

$$i(\alpha|D\beta) - i(D\alpha|\beta)$$
$$= \left(\alpha|c^j grad_{\delta_j}^S \beta\right) + \left(c^j grad_{\delta_j}^S \alpha|\beta\right)$$
$$= \left(\alpha|grad_{\delta_j}^S c^j \beta\right)$$
$$- \left(\alpha|c\left(grad_{\delta_j}^S dx^j\right)\beta\right)$$
$$+ \left(grad_{\delta_j}^S \alpha|c^j \beta\right)$$
$$= \delta_j(\alpha|c^j\beta)$$
$$- \left(\alpha|c\left(grad_{\delta_j}^S dx^j\right)\beta\right). \qquad (16)$$

We updated the Leibniz rule used for $grad^S$, and the autocorrelation of c^j because dx^j is a real local $1 - form$, form, and $grad^S$ contracts.

By duality, the map $\alpha \longmapsto (\alpha|c(\alpha)\beta)$, which takes $1 - forms$ to functions, defines a vector field $\mathbb{Z}_{\alpha\beta} - because$ $X(N)$ $= end_{c^{\infty}(N)}\left(B^1(N), C^{\infty}(N)\right)$ so the right side becomes:

$$\delta_j\left(dx^j(\mathbb{Z}_{\alpha\beta})\right)$$
$$- \left(grad_{\delta_j} dx^j\right)(\mathbb{Z}_{\alpha\beta})$$
$$= dx^j\left(grad_{\delta_j}\mathbb{Z}_{\alpha\beta}\right)$$
$$= div\mathbb{Z}_{\alpha\beta} \qquad (17)$$

Leibniz's rule was used for binary connection on $B^1(N)$ and on $X(N)$, respectively, thus:

$$(\alpha|D\beta) - (D\alpha|\beta)$$
$$= -i\ div\mathbb{Z}_{\alpha\beta} \qquad (18)$$

Which has integral zero.

*** Dirac self-actuator**

If G is a density-bounded operator in a Hilbert space $\mathcal{H}$, i adjacent to this domain:

$$G = \{\alpha \in \mathcal{H} : \exists \varkappa \in with\ \langle G\beta|\alpha\rangle = \langle \beta|\varkappa\rangle\ for\ all\ \beta \in DomainG\ \}$$

then $G^*\alpha = \varkappa$, so that the formula $\langle G\beta|\alpha\rangle = \langle \beta|G^*\alpha\rangle$ remains constant. If the form G is symmetric, then the $Domain\ G \subseteq DomainG^*$ with $G^* = G$ on $Domain\ G$, that is G^* is an extension of G to a wider domain than it. The other adjacent part $G^{**} = \bar{G}$ is called the closure of the form G (its identical operators are always in this closure), where the domain of the closure is:

$$Dom\ \bar{G} = \{\beta \in \mathcal{H} : \exists \alpha \in \mathcal{H}\ and\ asequence\ \{\beta_n\} \subset Dom\ G, such\ that\ \beta_n \to \beta\ and\ \beta_n \to \alpha\ \in \mathcal{H}\}$$

Then the graph of $\bar{G}$ in $\mathcal{H} \oplus \mathcal{H}$ is the closure of the graph of G.[15] And then, we put $\bar{G}\beta = \alpha$, when G is symmetric, we have:

$$Dom\ G \subseteq Dom\ \bar{G} \subseteq Dom\ G^*$$

Definition 4.1. We have that G is self-actuator if $G = G^*$, thus G is symmetric and closed and therefor we have G is essentially self-adjoint if it is symmetric and its closure $\bar{G}$ is self-adjoint. [16]

Theorem 4.2. let (N, g) be compact boundaryless Riemannian spin manifold. The Dirac operator D is essentially self-adjoint on its original domain S.

Proof . there is a natural number of norms on $Dom\ D$, given by:

$$|||\beta|||^2 = \|\beta\|^2 + \|D^*\beta\|^2 \qquad (19)$$

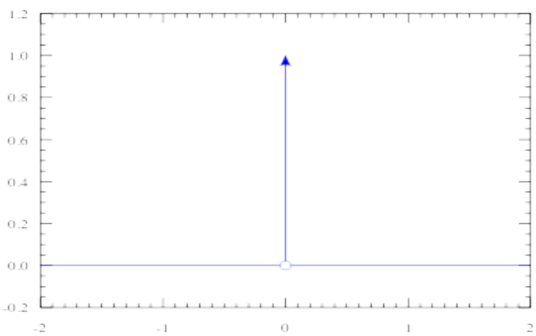
We have that $S = \Gamma(N, S)$ is dense in $Dom\ D^*$ for this norm. Using a finite partition of unity $f_1 + f_2 + \cdots + f_n = 1$ with each $f_i \in \mathcal{A}$ supported in chart domain U_i over which $S|u_i \to U_i$ is trivial, it is enough to show that any $f_i\alpha$, with $\alpha \in Dom\ D^*$, can be approximated in the $|||.||| - norm$ by elements of $\Gamma(U_i, S)$. Thus, we can suppose that $supp\ \alpha \subset U_i$, and regard $\alpha \in L^2(U_i, S)$ as a $2^n - tuple$ of functions $\alpha = \{\alpha_k\}$ with each $\alpha_k \in L^2(U_i, v_g)$ and given formula:

$$\langle D^*\alpha|\beta\rangle = \langle \alpha|D\beta\rangle = \int_N \left(\alpha|c^j\nabla^S_{\delta_j}\beta\right) = \int \left(\left(\alpha|c^j\nabla^S_{\delta_j}\beta\right) - \left(c^j\nabla^S_{\delta_j}\alpha|\beta\right)\right) v_g$$

$$= \int_N \left(-(c^j \alpha|\beta)(div\ \delta_j) - \left(c^j \nabla^S_{\delta_j} \alpha|\beta \right) \right) v_g \quad (20)$$

After an integration by parts, so that D^* is given by the formula:

$$D^* = -\left(\nabla^S_{\delta_j} + div\ \delta_j \right) c(dx^j)$$



Dirac operator delta function

As a vector-valued distribution on U_i, in particular, it is also a differential operator. So, if $\{h_i\}$ is a smooth delta-sequence, then for large enough i we can convolve both $\alpha\ and\ D^*\alpha$ with h_i, while remaining supported in U_i the convolution is defined after pulling back functions on the chart domain U_i to an fixed open subset of $\mathbb{R}^n$. Thus we find that $\alpha * h_i \to \alpha\ and\ D^*(\alpha * h_i) \to D^*\alpha\ in\ L^2(U_i, v_g)^{2^n}$, so that $|||\alpha * h_i - \alpha||| \to 0$. But the spinors $\alpha * h_i$ are smooth since the h_i are smooth, so we conclude that S is $|||.||| - dense$ in $Dom\ D^*$. We have $D^*(\alpha * h_i) = D(\alpha * h_i)$ since $S = Dom\ D$, so we have shown that α lies in $Dom\ \bar{D}$ and that $\bar{D}\alpha = D^*\alpha$. Thus $Dom\ \bar{D} = Dom\ D^*$, and it follows that $\bar{D} = D^{**} = D^*$, which establishes that $\bar{D}$ is self-adjoint. [16,17,20]

4.2 The Spectral growth of the Dirac operator

Since $D^2 = \Delta^S + \frac{1}{4} S$ and Δ^S is closely related to the Laplacian Δ on the (compact, boundaryless) Riemannian manifold (N, g), the general features of $sp(D)$ may be dedicated to those of $sp(\Delta)$.

We require two main properties of Laplacians on compact Riemannian manifolds without boundary. Recall that:

$$\Delta = -Gi_g\left(\nabla^{G*N \otimes G^*N} \circ \nabla^{G*N}\right) = -g^{ij}\left(\delta_j \delta_j - \Gamma^k_{ij}\delta_k\right) \quad (21)$$

is the local expression for the Laplacian (which depends on g through the Levi-Civita connection and g^{ij}). Thus Δ is a second order differential operator on $C^\infty(N)$.[17]

Corollary 4.3. Laplacian Δ has discrete point spectrum of finite multiplicity.

Proof: Since $(1 + \Delta)^{-1}$ is compact, its spectrum except for 0 consists only of eigenvalues of finite multiplicity. Therefore, the same is true of $(1+\Delta)$ and of Δ itself. We have:

$$sp(1+\Delta)^{-1} = \left\{\frac{1}{1+\lambda_0}, \frac{1}{1+\lambda_1}, \frac{1}{1+\lambda_2}, \ldots\right\} \qquad (22)$$

With

$\lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \cdots$ being the list of eigenvalues of Δ in increasing order.[6,18]These are counted with multiplicity, an eigenvalue of multiplicity i appears exactly i times on the list. This $\lambda_k \to \infty$, since $(1+\lambda_k)^{-1} \to \infty$

Corollary 4.4.

$$M_{|D|}(\lambda) \sim \frac{2^n \Omega_m}{n(2\pi)^m} \operatorname{vol}(N)\lambda^m, \quad as\ \lambda \to \infty$$

Example 4.5. Let $N = \mathbb{S}^2$, with $n = 2$, we have given:

$sp(D) = \left\{\pm\left(l+\frac{1}{2}\right) : l+\frac{1}{2} \in \mathbb{N}+\frac{1}{2}\right\}$,

With multiplicities $2l+1 = \{\pm k : k = 1,2,3,\ldots\}$

Therefore

$M_{|D|}(\lambda) = \sum_{1\leq k\leq\lambda} 4k = 2\lfloor\lambda\rfloor(\lfloor\lambda\rfloor+1) \sim 2\lambda^2$, as $\lambda \to \infty$

And $C_2 = \frac{\Omega_2}{2(2\pi)^2} = \frac{4\pi}{16\pi^2} = \frac{2\pi}{8\pi^2} = \frac{1}{4\pi}$ and $2C_2 = \frac{1}{2\pi}$

for spinors. Therefor $2C_2$ $Area(\mathbb{S}^2)\,\lambda^2 = 2\lambda^2$

$$Area(\mathbb{S}^2)\,\frac{1}{C_2} = 4\pi$$

The $2-sphere$ $\mathbb{S}^2$ form the knowledge of the circumference of the circle $\Omega_2 = 2\pi$ and the growth of the spectrum of the Dirac operator on $\mathbb{S}^2$.[18]

Example 4.6

Let $N = \mathbb{S}^1$, regarded as $\mathbb{S}^1 \simeq \frac{\mathbb{R}}{\mathbb{Z}}$ we parametrize the circle by the half-open interval $[0,1)$ rather than $[0,2\pi)$, say. Then $\mathcal{A} = C^\infty(\mathbb{S}^1)$ can be identified with periodic smooth functions on $\mathbb{R}$ with period 1and that:

$$\mathcal{A} \simeq \{f \in C^\infty(\mathbb{R}) : f(t+1) \equiv f(t)\} \qquad (23)$$

Since $C_1(\mathbb{R}) = C_1 \oplus C_{e_1}$ as a $\mathbb{Z}_{2-graded}$ algebra, we see that $\mathcal{B} = \mathcal{A}$ in this case, and since

$m = 1, n = 0$ and $2^{2n} = 1$, there is a trivial spin structure given by $S = \mathcal{A}$ itself. The charge conjugation is just $C = K$ where K mean complex conjugation of functions.[20] With the flat metric on the circle, the Dirac operator is just given as follows:

$$D = -i\frac{da}{dt}$$

where.

*** The torus uses Riemannian metric**

On the $2-torus\ \mathbb{T}^2=\frac{\mathbb{R}^2}{\mathbb{Z}^2}$, we use the Riemannian metric coming from the usual flat metric on $\mathbb{R}^2$. Thus, if we regard $\mathcal{A}=C^\infty(\mathbb{T}^2)$ as the smooth periodic function on $\mathbb{R}^2$ with $f(t,t^2)\equiv f(t+t^2)\equiv f(t,t^2+1)$ then (t,t^2) define local coordinates on $\mathbb{T}^2$, with respect to which all Christoffel symbols are zero, namely $\Gamma_{ij}^k=0$ and thus $\nabla=d$ represents the Levi-Civita connection on 1-forms.

$$\omega^1=\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad \omega^2=\begin{pmatrix}0&-i\\i&0\end{pmatrix},\quad \omega^2=\begin{pmatrix}1&0\\0&1\end{pmatrix}, \qquad (24)$$

we can write the charge conjugation operator as:

$$C=-i\,\omega^2K$$

where K again denotes complex conjugation. [5,17,19]

*** The Dirac operator on the sphere $\mathbb{S}^2$(the spinor bundle S on $\mathbb{S}^2$)**

Assume that the 2-dimensional sphere $\mathbb{S}^2$, with its usual orientation, $\mathbb{S}^2=\mathbb{C}\cup\{\infty\}\simeq\mathbb{C}P^1$. The usual spherical coordinates on $\mathbb{S}^2$ are:

$$P=(\sin(\theta)\cos(\emptyset),\sin(\theta)\sin(\emptyset),\cos(\theta))\in\mathbb{S}^2 \qquad (25)$$

The poles are $N=(0,0,1)\ \&\ S=(0,0,-1)$. Let $U_N=\mathbb{S}^2\setminus\{N\}, U_S=\mathbb{S}^2\setminus\{S\}$ be the two charts on $\mathbb{S}^2$. Consider the stereographic projections $p\mapsto z: U_N\longrightarrow\mathbb{C}, p\mapsto\xi: U_S\longrightarrow\mathbb{C}$ given by :

$$z=e^{-i\theta}\cot\frac{\theta}{2},\qquad \eta=e^{+i\emptyset}tan\frac{\theta}{2},$$

So that $\eta=\frac{1}{z}$ on $U_N\cap U_S$. Write

$$q=1+z\bar{z}=\frac{2}{1-\cos\theta},\ and\ q'=1+\eta\bar{\eta}=\frac{q}{z\bar{z}}$$

The sphere $\mathbb{S}^2$ has only the (trivial) spin structure $S=\Gamma(\mathbb{S}^2,S)$, where $S\to\mathbb{S}^2$ has rank tow. [20] Now $S=S^+\oplus S^-$, where $S^\pm\to\mathbb{S}^2$ are complex line bundles, and these may be nontrivial. We argue that $S^+\to\mathbb{S}^2$ is the line bundle coming from $\mathbb{S}^2\simeq\mathbb{C}P^1$. We know already that

$$\boldsymbol{S}^{**}\simeq\boldsymbol{S}\Leftrightarrow\boldsymbol{S}^*\simeq\boldsymbol{S}\Leftarrow(\boldsymbol{S}^+)^*\simeq\boldsymbol{S}^-$$

and the converse $\boldsymbol{S}^*\simeq\boldsymbol{S}\Rightarrow(\boldsymbol{S}^+)^*\simeq\boldsymbol{S}^-$ will hold provided we can show that $\boldsymbol{S}^\pm\longrightarrow\mathbb{S}^2$ are nontrivial line bundles. (Otherwise, $S^+$and S^- would each be self – dual, but we know that the only self - dual line bundle on $\mathbb{S}^2$ is the trivial one, since $\mathbb{H}^2\left((\mathbb{S}^2,\mathbb{Z})\simeq\mathbb{Z}\right)$. Consider now the line bundle $L\longrightarrow\mathbb{S}^2$, were

$L_{\mathcal{Z}} = \{(\lambda z_0, \lambda z_1) \in \mathbb{C}^2 : \lambda \in bC\}$,

$$if\ z = \frac{z_1}{z_1}, \quad L_\infty = \{(0, \lambda) \in \mathbb{C}^2 : \lambda \in \mathbb{C}\}.$$

In other words, $L_{\mathcal{Z}}$ is the complex line through the point $(1, z)$, for $z \in \mathbb{C}$. A particular local of L, defined over U_N is $\sigma_N(z) = \left(q^{-\frac{1}{2}}, zq^{-\frac{1}{2}}\right)$, which is normalized so that:

$(\sigma_N|\sigma_N) = q^{-1}(1 + \bar{z}z) = 1$ on U_N

this Hermitian pairing on $\Gamma(\mathbb{S}^2, L)$ comes from the standard scalar product on $\mathbb{C}^2$ each $L_{\mathcal{Z}}$ is a line in $\mathbb{C}^2$.

Let also $\sigma_S(\xi) = \left(\xi q^{l-\frac{1}{2}}, q^{l-\frac{1}{2}}\right)$ normalized so that $(\sigma_S|\sigma_S) = 1$ on U_S.

Now if $z \neq 0$, then

$$\sigma_S(z^{-1}) = \left(\frac{1}{z\sqrt{q^l}}, \frac{1}{\sqrt{q^l}}\right) = \left(\frac{\bar{z}}{z}\right)^{\frac{1}{2}} \sigma_N(z). \quad (26)$$

To avoid ambiguity, we state $\left(\frac{\bar{z}}{z}\right)^{\frac{1}{2}}$ means $e^{-i\theta}$, and $\left(\frac{\bar{z}}{z}\right)^{\frac{1}{2}}$ will mean $e^{+i\theta}$. [1,4,18]

A smooth section of L is given by two functions $\varphi_S^+(\zeta, \bar{\zeta})$ satisfying the relation.

$\varphi_S^+\left(\frac{\bar{z}}{z}\right)\sigma_N(z) = \varphi_S^+(\zeta, \bar{\zeta})\sigma_S(\zeta)$ on $U_N \cap U_S$.

Thus, we argue that:

$$\varphi_N^+\left(\frac{\bar{z}}{z}\right) = \left(\frac{\bar{z}}{z}\right)^{\frac{1}{2}} \varphi_S^+\left(\frac{z^{-1}}{}\bar{z}^{-1}\right)\ for\ z \neq 0 \quad (27)$$

And φ_N^+, φ_S^+ are regular at $z = 0$ or $\zeta = 0$ respectively. Likewise, a pair of smooth functions φ_N^-, φ_S^- on $\mathbb{C}$ in the dual bundle $L^* \to \mathbb{S}^2$ if and only if

$$\varphi_N^-\left(\frac{\bar{z}}{z}\right) = \left(\frac{\bar{z}}{z}\right)^{\frac{1}{2}} \varphi_S^-\left(\frac{z^{-1}}{}\bar{z}^{-1}\right)\ for\ z \neq 0$$

We claim now that we can identify $S^+ \simeq L^* = L^{-1}$ here the notation L^{-1} mean that (L^{-1}) is the inverse of (L) in the Picard group $H^2(\mathbb{S}^2, \mathbb{Z})$ that classifies $\mathbb{C}$ line bundles, so that a spinor in

$\mathbb{S} = \Gamma(\mathbb{S}^2, S)$ is given precisely by two pairs of smooth functions:

$$\begin{pmatrix} \varphi_N^-\left(\frac{\bar{z}}{z}\right) \\ \varphi_N^-\left(\frac{\bar{z}}{z}\right) \end{pmatrix} \text{on } U_N \qquad \begin{pmatrix} \varphi_S^+(\zeta, \bar{\zeta}) \\ \varphi_S^-(\zeta, \bar{\zeta}) \end{pmatrix} \text{on } U_S$$

Satisfying the above transformation rules. Since $S \otimes_A S^* \simeq End_A(S) \simeq \mathcal{B} \simeq A^\circ(\mathbb{S}^2)$ as A- module isomorphisms, it is enough to show that, as vector bundles,

$$A^\circ(\mathbb{S}^2) \simeq L^0 \oplus L^2 \oplus L^{-2} \oplus L^0,$$

Where $L^2 = L \otimes L$, $L^{-2} = L^* \otimes L^*$, and $L^0 = \mathbb{S}^2 \times \mathbb{C}$ is trivial line bundle, It is clear that $\mathcal{A}^0(\mathbb{S}^2) = C^\infty(\mathbb{S}^2) = \mathcal{A} = \Gamma(\mathbb{S}^2, L^0)$, and furthermore, $\mathcal{A}^2(\mathbb{S}^2) \simeq$

$\mathcal{A} = \Gamma(\mathbb{S}^2, L^0)$ since $\Lambda^2 T^*\mathbb{S}^2$ has a nonvanishing global section, namely the volume form $v = \sin\theta \, d\theta \wedge d\alpha$. [11,16]

With respect to the "round" metric on $\mathbb{S}^2$, namely,

$$g = d\theta^2 + sin^2\theta \, d\emptyset^2 = \frac{4}{a^2}(dx^1 \otimes dx^1 + dx^2 \otimes dx^2),$$

the pairs of 1-forms $\left\{\frac{dz}{q}, \frac{d\bar{z}}{q}\right\}$ *and* $\left\{-\frac{d\zeta}{q'}, -\frac{d\bar{\zeta}}{q'}\right\}$ are local bases for $\mathcal{A}^1(\mathbb{S}^2)$, over U_N and U_S respectively.

*** Spinor harmonics and the Dirac operator spectrum**

We introduce a set of special functions on $\mathbb{S}^2$ that are orthogonal basis normal to the rotors, in the same way that classical spherical harmonics Y_{im} yield an orthonormal basis of $L^2 - function.$ For functions, [2,6,19] l and m are integers, but the spinors are labelled by (half- odd-integers) in $\mathbb{Z} + \frac{1}{2}$. When expressed in our coordinates $(z, \bar{z})$, they are given as:-

$$l \in \left\{\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \dots\right\} = \mathbb{N} + \frac{1}{2} \text{ and } m \in \{-l, -l+1, \dots, l-1, l\} \quad (28)$$

Write

$$\boldsymbol{Y}^+_{\boldsymbol{lm}}(z,\bar{z}) = C_{lm} q^{-1} \sum_{r-s=m-\frac{1}{2}} \binom{l-\frac{1}{2}}{r} \binom{l+\frac{1}{2}}{S} z^r (-\bar{z})^S, \quad (29)$$

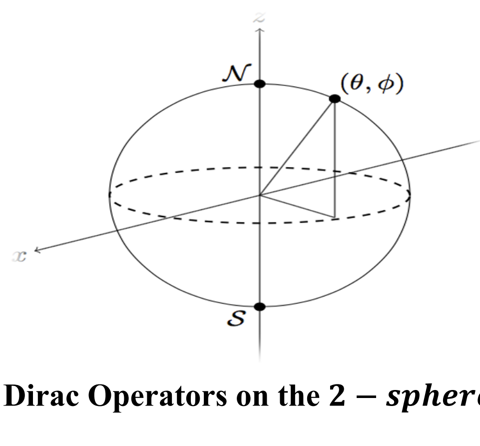
$$\boldsymbol{Y}^-_{\boldsymbol{lm}}(z,\bar{z}) = C_{lm} q^{-1} \sum_{r-s=m+\frac{1}{2}} \binom{l+\frac{1}{2}}{r} \binom{l-\frac{1}{2}}{S} z^r (-\bar{z})^S, \quad (30)$$

Where r, S are integers with $0 \le r \le l \mp \frac{1}{2}$ and $0 \le S \le l \pm \frac{1}{2}$ and

$$C_{im} = (-1)^{l-m} \sqrt{\frac{2l+1}{4\pi}} \sqrt{\frac{(l+m)!(l-m)!}{\left(l+\frac{1}{2}\right)! \left(l-\frac{1}{2}\right)!}} \quad (31)$$

*** Dirac Operators on the $\boldsymbol{2 - sphere}$**

We find that the projective units created are constrained across $C^* -$ algebra $A = C(\mathbb{S}^2)$ are of the form pA^k, where $p = (p_{ij})$ is an $k \times k$ matrix with elements in A, such that $p = (p^2 = p^*)$ is an orthogonal projector, whose rank is $tr\, p = p_{11} + \dots + p_{kk}$. To get module of rank one. It is enough to consider the case $k = 2$ of 2×2 matrices. [2,18,19]



Dirac Operators on the $\boldsymbol{2 - sphere}$

* Conclusion

The self-adroitness of the two-dimensional Dirac operator with Quantum dot and Lorentz-scalar δ-shell boundary conditions, on piecewise C^2 domains with finitely many corners. We investigate delta-interactions supported on surfaces satisfying certain weak topological conditions. We study the Dirac operator coupled on uniformly rectifiable surfaces. Under certain conditions on the coupling constants, we prove the self-jointness of the perturbed operator and we establish several spectral properties in the Lipschitz case. The main part of our paper consists of a detailed study of the problem on an infinite sector, where explicit computations can be made, we find the self-adjoint extensions for this case. Moreover, we result to other delta-shell interactions and derive several models of Dirac operators that give rise to the confinement phenomenon. We are concerned with the study of the pseudodifferential properties of operators associated with the Dirac operator with the boundary condition. For both models, we prove the existence of a unique self-adjoint realization whose domain is included in the Sobolev space $\mathcal{H} = L^2(N, K)$ the formal form domain of the free Dirac operator. The result is then translated to general domains by a coordinate transformation.

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