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On The Convergence and Stability of Semi-Analytic Technique Applying to Hyperbolic-Elliptic Systems of Conservation Laws

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Abstract

The main objective of our analysis is to expand the use of the Adomian-Rach decomposition method to process the mixed hyperbolic-elliptic system of conservation laws in gas fluid. The used method is based on decomposing the nonlinear terms by the Adomian polynomials and partitioning the given domain to avoid divergence as the time increases. The merged method presents a semi-analytic solution. A mixed hyperbolic-elliptic Cauchy-type problem is considered to examine our approach. Convergence and stability are also discussed numerically to show the effectiveness and efficiency of the method.

Keywords: Series solution; Numerical solution, Conservation laws; Huperbolic-elliptic system; Decomposition method.

* Introduction

Nonlinear partial differential equations are encountered in various fields of science. They can be used to describe a wide variety of physical phenomena ranging from gravitation to fluid dynamics. The notion of conservation laws plays an important role in the study of non-linear partial differential equations and systems which are of great importance in many areas of physics. The mathematical idea of conservation laws comes from the formulation of familiar physical laws of conservation of energy, conservation of momentum and so on.

The system of conservation laws is hyperbolic if the corresponding Jacobean matrix has real distinct eigenvalues and is elliptic if they are complex. The case study of this project is the systems of mixed hyperbolic-elliptic type. The mathematical ill-posedness of the system in the elliptic region reveals the physical fact that the state in the elliptic region is not stable, and it typically evolves into phase transitions. We show that our new proposed approach is stable inside the elliptic region.

As there are almost no general techniques that work for all such equations, several authors have devoted their attention to study the existence, uniqueness and approximate numeric-analytic solutions of this type of equations. Among these attempts are the finite difference method (Sod, 1978), the sine-Galerkin method (Alquran and Al-Khaled, 2011), variational iteration method (Raftari and Yildirim, 2012, Az-Zo'bi, 2014), homotopy analysis method (Hosseini et al., 2010), homotopy perturbation method (Mohyud-din et al., 2010) and the reduced differential transform method (Az-Zo'bi and Al dawoud, 2014, Az-Zo'bi, Al Dawoud and Marashdeh, 2015).

To construct approximate analytic solutions for singular mixed-type systems of conservation laws with different nonlinearities, the basic motivation of the study is to improve and extend the multi-stage Adomian-Rach decomposition method. The convergence and stability would be studied.

* **Methodology**

Recently, the Adomian-Rach decomposition method has been developed (Duan et al., 2015) to overcome the singularity and present numerical solutions of ordinary differential equations. It is based on combining series solution and decomposition method for solving nonlinear differential equations with Adomian polynomials for nonlinearities. This method avoids linearization, perturbation, discretization or any unrealistic assumptions which provides the solution in a rapid convergent series with easily computable components with the aid of MATHEMATICA software package.

The Adomian polynomials are not unique. Using the Taylor series expansion of analytic partial functions, a reliable new arrangement of Adomian polynomials were suggested (Az-Zo'bi and Al Khaled, 2010) to enforce many additional terms to the calculation processes that imply faster convergence. The relatively new formula for Adomian polynomials is considered in this work.

*** A first order partial differential equation of the form**

$$v_t + p(v)_x = 0,\ x \in \mathbb{R},\ t \in (0,T),\ (1)$$

is known as a scalar conservation law in one space dimension. $v(x,t)$ represents the conserved quantity, p is the flux function which is assumed to be analytic, x is the one-dimensional space variable, while t denotes the time. This type of equations often describes transport phenomena. The initial data are

$$v_t(x,0) = f(x)\ .(2)$$

The ADM expresses the solution $v(x,t)$ in the formal series

$$v(x,t) = \sum_{k=0}^{\infty} v_k(x) t^k \quad , \qquad (3)$$

where, A_k's are the Adomian polynomials (AP's) defined by

$$A_k(v_0,\ldots,v_k) = \frac{1}{k!}\left[\frac{d^k}{d\lambda^k} p\left(\sum_{i=0}^{\infty} v_i(x,t)\lambda^i\right)\right]_{\lambda=0}$$

, λ is a parameter, (4)

and,

$$p(v) = \sum_{k=0}^{\infty} A_k(v_0,\ldots,v_k) \quad . \qquad (5)$$

The Mathematica code for computing AP's is

$$For[k=0, k \le N, k++, \{A_k[x] = D\left[\frac{1}{k!} p\left[\sum_{i=0}^{k} \lambda^i v_i[x,t]\right], \{\lambda, k\}\right]$$

$$/.\lambda \to 0, Print\left["A"_k, "=", Expand\left[A_k[x]\right]\right]\}];$$

Consider the equally-spaced partition on the given domain $0 = t_0 < t_1 < t_2 < \cdots < t_N = T$, with the step-size $h = (b-a)/N$. The multi-stage ARDM states that the solution $v(x,t)$ on the subinterval $(t_m, t_{m+1}]$, $m = 0,1,\ldots,N-1$, is represented by the decomposition series

$$v(x,t) = \sum_{k=0}^{\infty} v_k(x)(t-t_m)^k \quad ,(6)$$

The nonlinear term $p(v)$ is decomposed in terms of solution coefficients a_n 's as

$$p(v) = \sum_{k=0}^{\infty} A_k(v_0,\ldots,v_k)(t-t_m)^k \quad .(7)$$

The remain steps of solution will carry out as mentioned in (Az-Zo'bi and Qousini, 2017). Also, the following theorem were proved. Theorem 1. The series solution given in Eq.(3) converges uniformly to the solution $v(x,t)$ of the IVP Eq.(1) on $|t-t_m| < R = \lim_{n\to\infty}\left(|v_{n+1}|/|v_n|\right) < \infty$.

*** Applications**

To verify the convergence and stability of the proposed algorithm to couple of mixed-type scalar conservation laws. The mixed-type Cauchy-problem with flux function (Holden and Risebro, 1990)

$$p(u,v)=\left(\frac{1}{2}\left(\frac{u^2}{2}+v^2\right)+v,u\,v\right), \quad (8)$$

subject to

$$v(x,0)=0.1x, u(x,0)=-1. \quad (9)$$

The elliptic region is determined and shown in Figure 1.

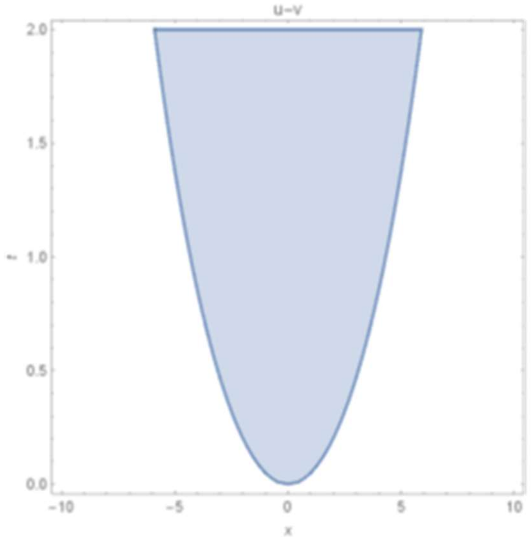
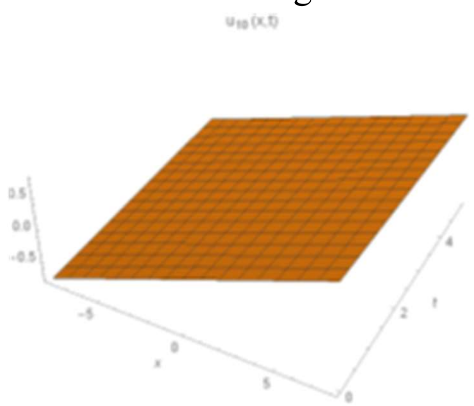


Figure 1. The elliptic region in the $u-v$ plane

Operating the Adomian-Rach decomposition algorithm, the 10th order series solutions are depicted in Figure 2. The corresponding absolute errors are shown in Figure 3.



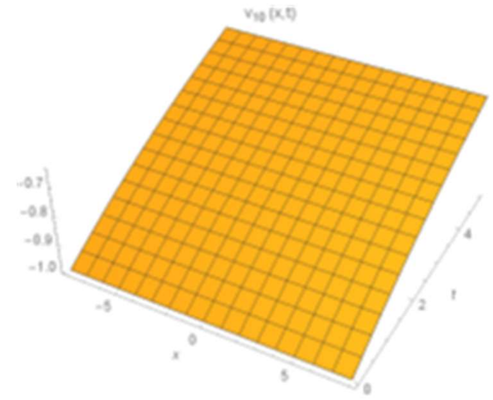
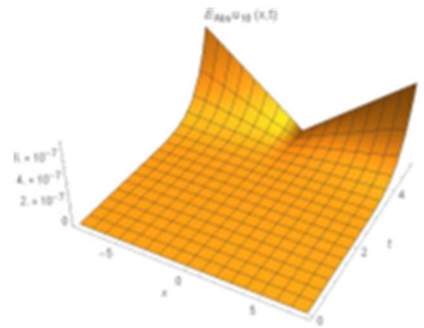


Figure 2. The approximate solutions of order 10.



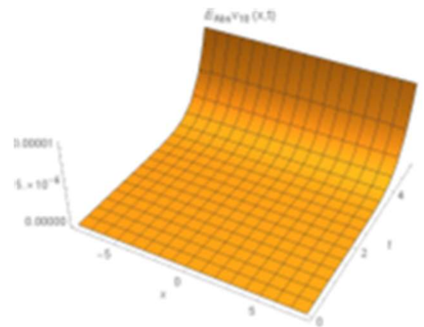


Figure 3. The related absolute errors of solutions in Figure 1.

The interaction parametric curve is shown in Figure 4. Also, Figure 5 illustrates the 2D approximate solutions for T={0, 0.5, 1, 1.5, 2, 2.5, 3}.

It is clear that the both are of high instability while transition through regions due to the oscillations. The solution keeps stability while the system crosses through different regions, which can be observed through small changes of solutions paths inside the elliptic region and preserving the path in the hyperbolic regions.

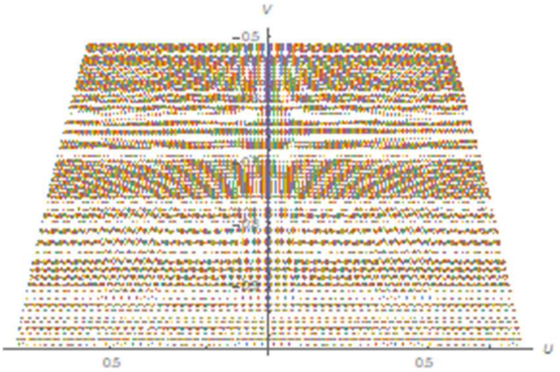


Figure 4. Parametric-numerical solution using ARDM.

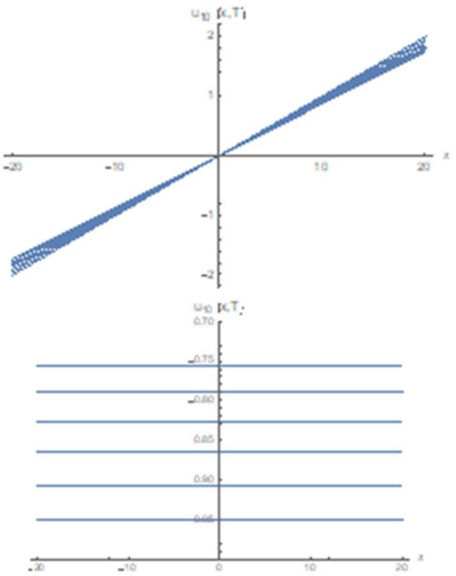


Figure 5. 2D plots of 10th-order approximate solutions.

* Conclusion

With unknown exact solutions for physical phenomena governed by systems of conservation laws, the obtained highly accurate quantitative approximate solutions via multi-stage modified decomposition method allow physicians to draw conclusions of systems under study in an efficient way. For mathematics, such studies make the pure mathematics more meaningful. The method doesn't need linearization, weak nonlinearity or perturbation. It is based on combining power series method and Adomian decomposition method by replacing nonlinearities with the corresponding Adomian polynomials expansions. While, in contrast to Adomian decomposition method and incomputable integrals for much nonlinearity, higher of series solution can be obtained easily using our modification. On the other hand, our technique overcomes the weakness and high complexity power series method in solving such problems.

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