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The Jordan homomorphisms of JC-algebra Tensor Product

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Abstract

An algebra homomorphism is a map that preserves the algebra operations. In this study, we consider the finite and infinite tensor products of JC-algebras. The main results as follows: we show how Jordan homomorphisms of the component JC-algebras can be combined to form Jordan homomorphisms of the JC-algebra tensor product. Also, we explain that the relation between the (C*-algebra) homomorphisms of C*-algebra tensor product and Jordan homomorphisms of JC-algebra tensor product by using universal enveloping C*-algebra.

Keywords: C*-algebra, JC-algebra, universal enveloping C*-algebra, homomorphism, tensor products of C*-algebras and JC-algebras.

*** Introduction**

A C*-algebra A is a Banach *-algebra which satisfied :-

$\|x^* x\| = (\|x\|)^2 \quad ,\forall x \in A$. For example : Let H be a complex Hilpert space . Then the space $B(H)$ of all bounded linear mappings on H with the norm : $\|T\| = \sup\{\|T\xi\| : \xi \in H ; \|\xi\| = 1\} \ ,\forall T \in B(H)$ is a C*-algebra.

Let H be a complex Hilpert space and $B(H)_{sa}$ is a self-adjoint part of the space $B(H)$. Then B is a JC-algebra if and only if it is a real Jordan Banach subalgebra of $B(H)_{sa}$.

Kadison and Ringrose (1986) characterized the (C*-algebra) homomorphisms of the tensor product of C*-algebras and the (C*-algebra) isomorphisms of the infinite tensor products of C*-algebras. Olsen an Stormer (1984) characterized the universal enveloping C*-algebra of a JC-algebra. Jamjoom (1994) defined the JC-algebra tensor product and she characterized the universal enveloping C*-algebra of the JC-algebra tensor product. Jamjoom (1997) defined the infinite tensor products of JC-algebras and she characterized the universal enveloping C*-algebra of the infinite products of JC-algebras.

Our purpose in this study is to give a characterization of Jordan homomorphisms of two cases:-

Case 1: The finite tensor products of JC -algebras.

Case 2 : The infinite tensor products of JC-algebras.

*** Methodology**

We use the mathematical proofs for obtaining desired results.

*** Results**

We define (C*-algebra) and Jordan homomorphisms as follows:-

Definition 1.

(i) Let A and B be C*-algebras , a mapping φ from A into B is described as a (C*-algebra) homomorphism if it is a homomorphism (that is, it is linear , multiplicative, that and carries the unit of A onto B) with the additional property that $\varphi(x^*)=\varphi(x)^*$ $,\forall x \in A$. If, further, φ is one -to- one, it is described as a (C*-algebra) isomorphism . (Kadison and Ringrose, 1983)

(ii) Let A and B be JC-algebras, a Jordan homomorphism $\varphi : A \xrightarrow{\text{into}} B$ is a linear mapping such that $\varphi(a \circ b)=\varphi(a)\circ\varphi(b)$ for each $a,b \in A$.

Jamjoom (1994) constructed the tensor products of JC-algebras as follows:-

Definition 2.

First: Given any pair of JC-algebras A , B and any C*-norm β on $C^*(A)\otimes C^*(B)$. We define $J(A\otimes B)$ to be the real Jordan subalgebra of $C^*(A)\otimes C^*(B)$ generated by $A\otimes B$.

Second : We define $JC\left(A\otimes_\beta B\right)$ to be the completion of $J(A\otimes B)$ in $C^*(A)\otimes_\beta C^*(B)$.

Third: It follows that $JC\left(A\otimes_\beta B\right)$ is the JC-subalgebra of $\left(C^*(A)\otimes_\beta C^*(B)\right)_{sa}$ generated by $A\otimes B$.

We call that $JC\left(A\otimes_\beta B\right)$ the JC-algebra tensor product of A and B with respect to β.

Jamjoom (1994) characterized the universal enveloping C*-algebra of the JC-algebra tensor product with respect to the minimum C*-cross-norm as follows:-

Corollary 3.

Let A and B be JC-algebras. Then:-

$C^*\left(JC\left(A \otimes_{\min} B\right)\right) = C^*(A) \otimes_{\min} C^*(B).$

We can see (Kadison and Ringrose, 1986, Section 11-4) for defining the infinite tensor products of C*-algebras. Jamjoom (1997) defined the infinite tensor products of JC-algebras as follows:-

Definition 4.

First: Let $\{A_i : i \in I\}$ be an infinite family of JC-algebras (not necessarily unitals) and let $\mathcal{F} = \{F \subseteq I : F$ is finite $\}$ and $\mathcal{F}$ is directed by the inclusion relation $\subseteq$.

Second : For each $F \in \mathcal{F}$, we can associate the JC-algebra tensor product $A_F = JC\left(\otimes_{\min} A_i\right)_{i \in F}$ of the finite family $\{A_i : i \in F\}$ which is the JC-subalgebra of $\left(\otimes_{\min} C^*(A_i)\right)_{i \in F}$ generated by $\left(\otimes A_i\right)_{i \in F}$.

Third : Now, we construct a direct system of JC-algebras as follows:-

If $F, G \in \mathcal{F}$ and $F \subseteq G$, then by Corollary 3 and the associativity of the tensor product, there is a natural Jordan isomorphism $\sigma_{GF} : JC\left(A_F \otimes_{\min} A_{G \setminus F}\right) \xrightarrow{onto} A_G$. Also, the equation :

$\alpha_{GF}(x) =$ strong limit $\sigma_{GF}\left(x \otimes v_\beta\right)$ $\left(x \in A_F\right)$

where $\{v_\beta\}$ is an approximate identity of $A_{G \setminus F}$, defines a Jordan homomorphism from A_F into A_G. Finally, if $F, G, H \in \mathcal{F}$ with $F \subseteq G \subseteq H$, then $\alpha_{HF} = \alpha_{HG} \circ \alpha_{GF}$, hence the family $\{A_F : F \in \mathcal{F}\}$ with the Jordan homomorphism α_{GF} is a directed system of JC-algebras .

Fourth : The JC-direct limit, say A of $\{A_F, \alpha_{GF}\}$ exists and is a JC-algebra , called the tensor product of the infinite family $\{A_i : i \in I\}$ of JC-algebras, and is denoted by $JC\left(\otimes_{\min} A_i\right)_{i \in I}$ .In other word,

$A = \xrightarrow{\lim} A_F = JC\left(\otimes_{\min} A_i\right)_{i \in I}$.

*** Theorem 5. (Jamjoom, 1997)**

Let $\{A_i : i \in I\}$ be the family of JC-algebras. Then

$C^*\left(JC\left(\otimes_{\min} A_i\right)_{i \in I}\right) = \left(\otimes_{\min} C^*(A_i)\right)_{i \in I}$

Now, Kadison and Ringrose (1986) characterized (C*-algebra) homomorphisms of the C*-algebra tensor product and (C*-algebra) isomorphisms of infinite -tensor-product C*-algebras as follows:

*** Proposition 6.**

Suppose that, for $i = 1,\ldots,n \ (n \in \bullet)$, A_i and B_i are C*-algebras, and $\varphi_i : A_i \xrightarrow{\text{into}} B_i$ is a (C*-algebra) homomorphism. Then there is a (C*-algebra) homomorphism

$$\varphi : A_1 \otimes_{\min} \cdots \otimes_{\min} A_n \xrightarrow{\text{into}} B_1 \otimes_{\min} \cdots \otimes_{\min} B_n$$

, uniquely determined by the condition:

$$\varphi(x_1 \otimes \cdots \otimes x_n) = \varphi_1(x_1) \otimes \cdots \otimes \varphi_n(x_n) \quad (x_1 \in$$

*** Remark 7.**

The (C*-algebra) homomorphism φ occurring in proposition 6 is denoted by $\varphi_1 \otimes_{\min} \cdots \otimes_{\min} \varphi_n$.

*** Proposition 8.**

Suppose that $\{A_i : i \in I\}$ and $\{B_i : i \in I\}$ are families of C*-algebras, $A = (\otimes_{\min} A_i)_{i \in I}$ and $B = (\otimes_{\min} B_i)_{i \in I}$. Let $A(i)$ denote the canonical image of A_i in A , $B(i)$ that of B_i in B .

(i) If A_i is (C*-algebra) isomorphic to B_i for each $i \in I$,then A is (C*-algebra) isomorphic to B .

(ii) If θ_i is a (C*-algebra) isomorphism from $A(i)$ onto $B(i)$ for each $i \in I$,there is a (C*-algebra) isomorphism θ from A onto B , such that for each $i \in I$.

*** Remark 9.**

The (C*-algebra) isomorphism θ occurring in proposition 8(ii) is denoted by $(\otimes_{\min} \theta_i)_{i \in I}$.

*** Discussion and Conclusions**

Now we show how Jordan homomorphisms of component JC-algebras A_1 and A_2 can be combined to form Jordan homomorphisms of $JC(A_1 \otimes_{\min} A_2)$ as follows:-

*** Proposition 10.**

Suppose that, for $i = 1,2$, A_i and B_i are JC-algebras, and $\varphi_i : A_i \xrightarrow{\text{into}} B_i$ is a Jordan homomorphism. Then there is a Jordan homomorphism $\varphi : JC(A_1 \otimes_{\min} A_2) \xrightarrow{\text{into}} JC(B_1 \otimes_{\min} B_2)$, uniquely determined by the condition:-

$$\varphi(a_1 \otimes a_2) = \varphi_1(a_1) \otimes \varphi_2(a_2) \quad (a_1 \in A_1, a_2 \in A_2),$$

(see diagram (1)) .

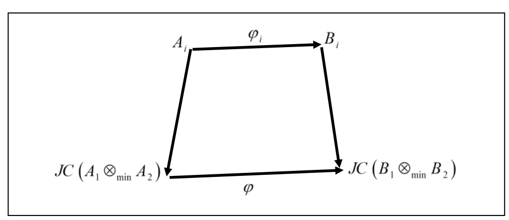

Diagram (1) .

Proof. Suppose that for $i = 1,2$, $\varphi_i : A_i \xrightarrow{\text{into}} B_i$ is a Jordan homomorphism. Since

A_i and B_i are JC-algebras, then the enveloping C*-algebras $C^*(A_i)$ and $C^*(B_i)$

exist and are unique by (Olsen an Stormer, 1984, Theorem 7.1.8) . Hence $\varphi_i : A_i \xrightarrow{\text{into}} B_i$ extends to a (C*-algebra) homomorphism $\varphi_i^* : C^*(A_i) \xrightarrow{\text{into}} C^*(B_i)$ by (Olsen an Stormer, 1984, Theorem 7.1.8) . Now from Corollary 3, we have

$$C^*(JC(A_1 \otimes_{\min} A_2)) = C^*(A_1) \otimes_{\min} C^*(A_2),$$
$$C^*(JC(B_1 \otimes_{\min} B_2)) = C^*(B_1) \otimes_{\min} C^*(B_2).$$

Hence by Proposition 6, there is a (C*-algebra) homomorphism

$$\varphi^* : C^*(A_1) \otimes_{\min} C^*(A_2) \xrightarrow{\text{into}} C^*(B_1) \otimes_{\min} C$$

, uniquely determined by the condition :

$$\varphi^*(x_1 \otimes x_2) = \varphi_1^*(x_1) \otimes \varphi_2^*(x_2) \quad (x_1 \in C^*(A_1), x_2 \in C^*(A_2)).$$

Then the restriction φ, (say), of φ^* to $JC(A_1 \otimes_{\min} A_2)$ has the following :

(i) $\varphi = \varphi^* \big| JC(A_1 \otimes_{\min} A_2)$ is a Jordan homomorphism from $JC(A_1 \otimes_{\min} A_2)$ to $JC(B_1 \otimes_{\min} B_2)$ because $JC(A_1 \otimes_{\min} A_2)$ is a JC-subalgebra of $C^*(A_1) \otimes_{\min} C^*(A_2)$ and

$$\varphi(JC(A_1 \otimes_{\min} A_2)) \subseteq JC(\varphi_1(A_1) \otimes_{\min} \varphi_2(A_2)) \subseteq JC(B_1 \otimes_{\min} B_2),$$

(ii) $\varphi(a_1 \otimes a_2) = \varphi_1(a_1) \otimes \varphi_2(a_2) \quad (a_1 \in A_1, a_2 \in A_2).$ +

*** Remark 11.**

The Jordan homomorphism φ occurring in proposition 10 is denoted by $\varphi_1 \otimes_{\min} \varphi_2$. Note that if A, B and C are JC-algebras, then

$$C^*(JC(A \otimes_{\min} B) \otimes_{\min} C) = C^*(JC(A \otimes_{\min} B)) \otimes_{\min} C^*(C)$$
$$= C^*(A) \otimes_{\min} C^*(B) \otimes_{\min} C^*(C).$$

Therefore $JC(JC(A \otimes_{\min} B) \otimes_{\min} C)$ is the JC-algebra $JC(A \otimes_{\min} B \otimes_{\min} C)$ generated by $A \otimes B \otimes C$ in $C^*(A) \otimes_{\min} C^*(B) \otimes_{\min} C^*(C)$. Then we have the following:-

*** Corollary 12.**

Suppose that, for $i = 1, \ldots, n$ $(n \in \bullet)$, A_i and B_i are JC-algebras, and $\varphi_i : A_i \xrightarrow{\text{into}} B_i$ is a Jordan homomorphism. Then there is a Jordan homomorphism

$$\varphi : JC(A_1 \otimes_{\min} \cdots \otimes_{\min} A_n) \xrightarrow{\text{into}} JC(B_1 \otimes_{\min} \cdots \otimes_{\min} B_n)$$

, uniquely determined by the condition:-

$$\varphi(a_1 \otimes \cdots \otimes a_n) = \varphi_1(a_1) \otimes \cdots \otimes \varphi_n(a_n) \quad (a_1 \in A_1, \ldots, a_n \in A_n).$$

We now consider certain Jordan homomorphisms of infinite-tensor-product JC-algebras as follows:

*** Proposition 13.**

Suppose that $\{A_i : i \in I\}$ and $\{B_i : i \in I\}$ are infinite families of JC-algebras. Then, if $\theta_i : A_i \xrightarrow{\text{into}} B_i$ is a Jordan homomorphism for each $i \in I$, then there is a Jordan homomorphism

$$\theta : JC\left(\otimes_{\min} A_i\right)_{i\in I} \xrightarrow{\text{into}} JC\left(\otimes_{\min} B_i\right)_{i\in I}$$

such that $\theta | A_i = \theta_i \ (i \in I)$

(see diagram 2) .

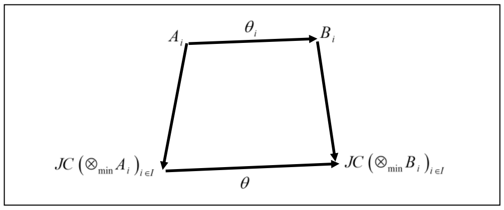

Diagram (2) .

Proof. Suppose that $i \in I$ arbitrary and $\theta_i : A_i \xrightarrow{\text{into}} B_i$ is a Jordan homomorphism. Since A_i and B_i are JC-algebras, then the enveloping C*-algebras $C^*(A_i)$ and $C^*(B_i)$ exist and are unique by (Olsen an Stormer, 1984, Theorem 7.1.8) . Hence $\theta_i : A_i \xrightarrow{\text{into}} B_i$ extends to a (C*-algebra) homomorphism $\theta_i^* : C^*(A_i) \xrightarrow{\text{into}} C^*(B_i)$ by (Olsen an Stormer, 1984, Theorem 7.1.8) . Now because $C^*(A_i)$ and $C^*(B_i)$ are C*-algebras. Then we can use Proposition 8(i) as follows:-

By Theorem 5, we have:-

$$C^*\left(JC\left(\otimes_{\min} A_i\right)_{i\in I}\right) = \left(\otimes_{\min} C^*\left(A_i\right)\right)_{i\in I},$$

$$C^*\left(JC\left(\otimes_{\min} B_i\right)_{i\in I}\right) = \left(\otimes_{\min} C^*\left(B_i\right)\right)_{i\in I}.$$

Hence by Proposition 8(I), there is a (C*-algebra) homomorphism

$$\theta^* : \left(\otimes_{\min} C^*\left(A_i\right)\right)_{i\in I} \xrightarrow{\text{into}} \left(\otimes_{\min} C^*\left(B_i\right)\right)_{i\in I}$$

such that $\theta^* | C^*(A_i) = \theta_i^*$ for each $i \in I$. Then the restriction θ , (say), of θ^* to $JC\left(\otimes_{\min} A_i\right)_{i\in I}$ has the following:

(i) $\theta = \theta^* | JC\left(\otimes_{\min} A_i\right)_{i\in I}$ is a Jordan homomorphism from $JC\left(\otimes_{\min} A_i\right)_{i\in I}$ into $JC\left(\otimes_{\min} B_i\right)_{i\in I}$, because $JC\left(\otimes_{\min} A_i\right)_{i\in I}$ is a JC-subalgebra of $\left(\otimes_{\min} C^*\left(A_i\right)\right)_{i\in I}$ and

$$\theta\left(JC\left(\otimes_{\min} A_i\right)_{i\in I}\right) \subseteq JC\left(\otimes_{\min} \theta_i\left(A_i\right)\right)_{i\in I} \subseteq JC\left(\otimes_{\min} B_i\right)_{i\in I}$$

(ii) $\theta | A_i = \theta_i \ (i \in I)$. +

* Recommendations

We can use the Propositions 10 and 13 for any problems about Jordan homomorphisms of the finite tensor products of JC -algebras and the infinite tensor products of JC-algebras.

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